THEORETICAL & COMPUTATIONAL CONSIDERATIONS OF STURM-LIOUVILLE SYSTEMS

Abstract

A common exercise encountered while studying analysis is to decompose a certain object in terms of a given basis. In particular, we focus on the representation of a given function $f \in L^2[a,b]$ by an infinite series involving functions from a certain basis of the function space. The idea of representations by series are often encountered while modelling and solving boundary value problems in ordinary and partial differential equations. To begin with, we discuss representations by Fourier series, which are certain types of infinite series constituting multi-angle sinusoidals. In Chapter 1, we present some theory on Fourier series, discuss its convergence and also go through some important numerical aspects of Fourier series approximations.

Once it is established that the set of multi-angle sinusoidals, i.e.,

$$\mathcal{F} = \{1, \sin nx, \cos nx \mid \text{for } n \text{ running over } \mathbb{N}\},$$

building blocks of a Fourier series, forms a basis of $L^2[-\pi, \pi]$, we see Fourier series from the perspective of regular SL systems. Note that, $\mathcal{F}$ constitutes eigenfunctions of a regular SL system. Hence, gradually in Chapter 2 we study the theory of Sturm-Liouville systems, in an attempt to generalise the concept of a Fourier series to an SL series. While the beginning of Chapter 2 mainly discusses types of SL systems and some results on eigenfunctions of SL systems, the main crux of it is to present the Oscillation Theory.

The discussion is then naturally carried forward in Chapter 3, where first of all existence of a sequence of eigenfunctions of any regular SL system is presented. Chapter 3 then discusses some theory on the asymptotic behaviour of eigenfunctions and distributions of eigenvalues of regular SL systems, eventually leading to establish the completeness of the set of eigenfunctions in the function space $L^2_{\rho(x)}[a,b]$.

Next, in Chapter 4, we broaden our persepective of looking at SL series approximations and hence consider concepts from approximation theory. For instance, one can formulate and present a suitable approximation problem keeping in mind the associated geometrical properties of a Fourier series approximation. Similarly from an application point of view, there exists important approximation problems which are solved by Chebyshev Polynomials. Finally, we present numerical methods for performing Chebyshev approximations efficiently. At a glimpse, unlike the numerical experiments on Fourier series approximations where the coefficients were computed using numerical integration, we discuss discrete computation of coefficients in Chebyshev series and fast numerical evaluation of Chebyshev series.